The conjecture on distance-balancedness of generalized Petersen graphs holds when internal edges have jumps 3 or 4

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Abstract

A connected graph G with $\mathrm{diam}(G) \geq \ell$ is ℓ-distance-balanced if $|W_{xy}| = |W_{yx}|$ for every $x, y \in V(G)$ with $d_G(x, y) = \ell$, where W_{xy} is the set of vertices of G that are closer to x than to y. Miklavič and Šparl conjectured that if n is not very small with respect to k, then the generalized Petersen graph $GP(n, k)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n, k))$. In the seminal paper, the conjecture was verified for $k = 2$. In this paper we prove that the conjecture holds for $k = 3$ and for $k = 4$.

Key words: distance-balanced graph; ℓ-distance-balanced graph; generalized Petersen graph; diameter

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1 Introduction

Let $G = (V(G), E(G))$ be a connected graph and $u, v \in V(G)$. The set of vertices that are closer to u than to v (with respect to the standard shortest-path distance $d_G(u,v)$) is denoted by W_{uv}. When $|W_{uv}| = |W_{vu}|$ holds, the vertices u and v are called *balanced*, and when every pair of adjacent vertices is balanced, G is called *distance-balanced*. Distance-balanced graphs were first considered in [11], the term "distance-balanced" was coined in [13]. For a number of reasons, both theoretical and applied, the distance-balanced graphs received a lot of attention, see [1,3–8,12, 15–17,20,22]. We should also mention in passing that distance-balanced graphs can be equivalently described as the graphs whose Mostar index (see [2]) equals 0.

More generally, let $\ell \in [\text{diam}(G)] = \{1, 2, \ldots, \text{diam}(G)\}$, where $\text{diam}(G)$ is the diameter of G. Then G is called *ℓ-distance-balanced* [9] if each pair of vertices $u, v \in V(G)$ with $d_G(u,v) = \ell$ is balanced. For a study of 2-distance-balanced graphs see [10] and for several results of ℓ-distance-balanced graphs see [14,21].

This paper is about the distance-balancedeness of the generalized Petersen graphs. The interest in these graphs was already shown in [13] where it was conjectured that for any integer $k \geq 2$, there exists a positive integer n_0 such that $GP(n,k)$ is not distance-balanced for every $n \geq n_0$. The validity of the conjecture has been demonstrated in [22]. Interest in the distance-balancedeness of the generalized Petersen graphs continued in [18,21]. In [18] it was proved that $GP(n,k)$ is $\text{diam}(GP(n,k))$-distance-balanced as soon as n is large relative to k, more precisely, the following theorem was proved.

Theorem 1. [18] *If n and k are integers, where $3 \leq k < n/2$, and*

$$n \geq \begin{cases} 8; & k = 3, \\ 10; & k = 4, \\ \frac{k(k+1)}{2}; & k \text{ is odd and } k \geq 5, \\ \frac{k^2}{2}; & k \text{ is even and } k \geq 6, \end{cases}$$

then $GP(n,k)$ is $\text{diam}(GP(n,k))$*-distance-balanced.*

On the other hand, Miklavič and Špar posed the following:

Conjecture 2. [21] *Let $k \geq 2$ be an integer and let*

$$n_k = \begin{cases} 11; & k = 2, \\ (k+1)^2; & k \text{ odd}, \\ k(k+2); & k \geq 4 \text{ even}. \end{cases}$$

Then for any $n > n_k$, the graph $GP(n,k)$ is not ℓ-distance-balanced for any $1 \leq \ell <$ $\text{diam}(GP(n,k))$*. Moreover, n_k is the smallest integer with this property.*

In [21], Conjecture 2 was verified for $k = 2$. In this paper, we prove that Conjecture 2 holds true for $k = 3$ and for $k = 4$ by establishing the following results.

Theorem 3. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n,3))$. Moreover, 16 is the smallest integer with this property.*

Theorem 4. *For any $n > 24$, the generalized Petersen graph $GP(n,4)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n,4))$. Moreover, 24 is the smallest integer with this property.*

To prove these two theorems, it suffices to prove the first assertion of each of them. With these results in hand, the facts that 16 is the smallest integer in Theorem 3 and that 24 is the smallest integer in Theorem 4, follow by computer experiments presented in [21, Table 1].

Full proofs of Theorems 3 and 4 are very long and repetitive. We therefore present in the next two sections only selected, representative cases. Complete proofs however can be found in [19]. Their lengths suggest that it would be difficult to generalize our approach to $k > 4$. We conclude the paper by suggesting a problem in Section 4.

To conclude the introduction recall that the *generalized Petersen graph* $GP(n,k)$, $n \geq 3$, $1 \leq k < n/2$, is defined by

$$V(GP(n,k)) = \{u_i : \ i \in \mathbb{Z}_n\} \cup \{v_i : \ i \in \mathbb{Z}_n\},$$
$$E(GP(n,k)) = \{u_i u_{i+1} : \ i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} : \ i \in \mathbb{Z}_n\} \cup \{u_i v_i : \ i \in \mathbb{Z}_n\}.$$

2 Sketch proof of Theorem 3

As mentioned in the introduction, it suffices to prove that for any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n,3))$. We split the argument into the cases $\ell = 1$, $\ell = 2$, and $3 \leq \ell < \mathrm{diam}(GP(n,3))$ to be respectively covered by Propositions 5, 6, and 7.

Proposition 5. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not 1-distance-balanced.*

Proof. Since $d_{GP(n,3)}(u_0, v_0) = 1$, it suffices to prove that $|W_{u_0v_0}| < |W_{v_0u_0}|$. We divide the discussion into six cases based on $n \bmod 6$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other five cases are given in [19].

Let $n = 6m$, where $m \geq 3$. By symmetry, it suffices to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$. Then the following holds.

- If $1 \le t \le m$, then $d(u_0, u_{3t}) = 2 + t$ and $d(v_0, u_{3t}) = 1 + t$.
- If $1 \le t \le m$, then $d(u_0, v_{3t}) = 1 + t$ and $d(v_0, v_{3t}) = t$.
- If $1 \le t < m$, then $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_0, u_{3t+1}) = 2 + t$.
- If $0 \le t < m$, then $d(u_0, v_{3t+1}) = 2 + t$ and $d(v_0, v_{3t+1}) = 3 + t$.
- If $1 \le t < m$, then $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_0, u_{3t+2}) = 3 + t$.
- If $0 \le t < m$, then $d(u_0, v_{3t+2}) = 3 + t$ and $d(v_0, v_{3t+2}) = 4 + t$.
- $d(u_0, u_1) = 1$ and $d(v_0, u_1) = 2$.
- $d(u_0, u_2) = 2$ and $d(v_0, u_2) = 3$.

In the above consideration, we have $2m+2$ vertices from $W_{u_0v_0}$ and $4m-2$ vertices from $W_{v_0u_0}$. Since we have considered only the vertices u_i and v_i with $1 \le i \le \frac{n}{2}$, there are in total twice as many vertices, except that u_{3m} and v_{3m} are considered twice (and they lie in $W_{v_0u_0}$). Since clearly $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$, we conclude that

$$|W_{u_0v_0}| = 2(2m+2) + 1 = 4m + 5,$$
$$|W_{v_0u_0}| = 2(4m-2) + 1 - 2 = 8m - 5.$$

Because $m \ge 3$, we indeed have $|W_{u_0v_0}| < |W_{v_0u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 6m + 1$, $m \ge 3$, then $|W_{u_0v_0}| = 4m + 3$ and $|W_{v_0u_0}| = 8m - 3$.
- If $n = 6m + 2$, $m \ge 3$, then $|W_{u_0v_0}| = 4m + 4$ and $|W_{v_0u_0}| = 8m$.
- If $n = 6m + 3$, $m \ge 3$, then $|W_{u_0v_0}| = 4m + 7$ and $|W_{v_0u_0}| = 8m - 1$.
- If $n = 6m + 4$, $m \ge 3$, then $|W_{u_0v_0}| = 4m + 6$ and $|W_{v_0u_0}| = 8m + 2$.
- If $n = 6m + 5$, $m \ge 2$, then $|W_{u_0v_0}| = 4m + 5$ and $|W_{v_0u_0}| = 8m + 3$.

Note that if $n = 6m+2$, $m \ge 3$, then $|W_{u_0v_0}| + |W_{v_0u_0}| < |V(GP(n,3))|$, the reason is that $d(u_0, v_{3(m-1)+1}) = m + 1 = d(v_0, v_{3(m-1)+1})$. Anyhow, in each case we have $|W_{u_0v_0}| < |W_{v_0u_0}|$. $\square$

Proposition 6. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not 2-distance-balanced.*

Proof. Since $d_{GP(n,3)}(u_0, v_{-3}) = 2$, it suffices to prove that $|W_{u_0v_{-3}}| < |W_{v_{-3}u_0}|$. We divide the discussion into the six cases based on n mod 6, and for transparency and non-replication purposes, present only the first case in detail. Details for the other five cases are given in [19].

Firstly we consider vertices $v_{-1}, v_{-2}, u_{-1}, u_{-2}$.

- $d(u_0, v_{-1}) = 2$ and $d(v_{-3}, v_{-1}) = 4$.
- $d(u_0, v_{-2}) = d(v_{-3}, v_{-2}) = 3$.
- $d(u_0, u_{-1}) = 1$ and $d(v_{-3}, u_{-1}) = 3$.
- $d(u_0, u_{-2}) = d(v_{-3}, u_{-2}) = 2$.

So $u_{-1}, v_{-1} \in W_{u_0v_{-3}}$ and no vertex of $\{v_{-1}, v_{-2}, u_{-1}, u_{-2}\}$ is in $W_{v_{-3}u_0}$.

Next we consider vertices v_i where $0 \le i < n-3$ and u_j where $1 \le j \le n-3$. Let $n = 6m$ where $m \ge 3$. Note that $v_{-3} = v_{n-3}$ and $n - 3 = 6m - 3 = 3(2m-1)$.

- If $0 \le t \le m-1$, then $d(u_0, v_{3t}) = d(v_{6m-3}, v_{3t}) = 1 + t$.
  If $m \le t < 2m-1$, then $d(v_{6m-3}, v_{3t}) = 2m-1-t$ and $d(u_0, v_{3t}) > d(v_{6m-3}, v_{3t})$.
- If $0 \le t \le m-1$, then $d(u_0, v_{3t+1}) = 2 + t$ and $d(u_0, v_{3t+1}) < d(v_{6m-3}, v_{3t+1})$.
  If $m \le t < 2m-1$, then $d(u_0, v_{3t+1}) = d(v_{6m-3}, v_{3t+1}) = 2m - t + 2$.
- If $0 \le t \le m-2$, then $d(u_0, v_{3t+2}) = 3 + t$ and $d(u_0, v_{3t+2}) < d(v_{6m-3}, v_{3t+2})$.
  If $m-1 \le t < 2m-1$, then $d(u_0, v_{3t+2}) = d(v_{6m-3}, v_{3t+2}) = 2m - t + 1$.
- If $1 \le t \le m-1$, then $d(u_0, u_{3t}) = d(v_{6m-3}, u_{3t}) = 2 + t$.
  If $m \le t \le 2m-1$, then $d(v_{6m-3}, u_{3t}) = 2m - t$ and $d(u_0, u_{3t}) > d(v_{6m-3}, u_{3t})$.
- If $1 \le t \le m-1$, then $d(u_0, u_{3t+1}) = d(v_{6m-3}, u_{3t+1}) = 3 + t$.
  If $m \le t < 2m-1$, then $d(v_{6m-3}, u_{3t+1}) = 2m - t + 1$ and $d(u_0, u_{3t+1}) > d(v_{6m-3}, u_{3t+1})$.
- If $1 \le t \le m-2$, then $d(u_0, u_{3t+2}) = d(v_{6m-3}, u_{3t+2}) = 4 + t$.
  If $m-1 \le t < 2m-1$, then $d(v_{6m-3}, u_{3t+2}) = 2m - t$ and $d(u_0, u_{3t+2}) > d(v_{6m-3}, u_{3t+2})$.
- $d(u_0, u_1) = 1$, $d(v_{6m-3}, u_1) = 2m + 1$, $d(u_0, u_2) = 2$, $d(v_{6m-3}, u_2) = 2m$.

Note that $u_0 \in W_{u_0v_{6m-3}}$ and $v_{6m-3} \in W_{v_{6m-3}u_0}$. Combined with the above discussion we get $|W_{u_0v_{6m-3}}| = 2m+4$ and $|W_{v_{6m-3}u_0}| = 4m-1$. Because $m \ge 3$, we can conclude that $|W_{u_0v_{6m-3}}| < |W_{v_{6m-3}u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 6m+1$, $m \geq 3$, then $|W_{u_0v_{-3}}| = 2m+4$ and $|W_{v_{-3}u_0}| = 4m+2$.
- If $n = 6m+2$, $m \geq 3$, then $|W_{u_0v_{-3}}| = 2m+3$ and $|W_{v_{-3}u_0}| = 4m+1$.
- If $n = 6m+3$, $m \geq 3$, then $|W_{u_0v_{-3}}| = 2m+6$ and $|W_{v_{-3}u_0}| = 4m+3$.
- If $n = 6m+4$, $m \geq 3$, then $|W_{u_0v_{-3}}| = 2m+4$ and $|W_{v_{-3}u_0}| = 4m+2$.
- If $n = 6m+5$, $m \geq 2$, then $|W_{u_0v_{-3}}| = 2m+5$ and $|W_{v_{-3}u_0}| = 4m+5$.

In each case we have $|W_{u_0v_{-3}}| < |W_{v_{-3}u_0}|$. □

Proposition 7. *For any $n > 16$, the generalized Petersen graph $GP(n,3)$ is not ℓ-distance-balanced for any $3 \leq \ell < \operatorname{diam}(GP(n,3))$.*

Proof. For a given fixed n, we set $D = \operatorname{diam}(GP(n,3))$.

For any $3 \leq \ell < D$, we first show that there exists v_j such that $d(u_0, v_j) = \ell$, where $6 \leq j \leq n/2$. From [18] we recall that there exists j^* such that $d(u_0, u_{j^*}) = D$.

If $n = 6m$ ($m \geq 3$) or $n = 6m+1$ ($m \geq 3$), then we know from [18] that $j^* = 3(m-1)+2$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+2}) = s+3$, where $2 \leq s \leq m-1$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+2$ ($m \geq 3$) or $n = 6m+3$ ($m \geq 3$), then from [18] we know that $j^* = 3m+1$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+1}) = s+2$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+4$ ($m \geq 3$), then from [18] we know that $j^* = 3m+2$ and $D = d(u_0, u_{j^*}) = m+4$. Note that $d(u_0, v_{3s+2}) = s+3$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

If $n = 6m+5$ ($m \geq 2$), then (again by [18]) $j^* = 3m+1$ and $D = d(u_0, u_{j^*}) = m+3$. Note that $d(u_0, v_{3s+1}) = s+2$, where $2 \leq s \leq m$, and $d(u_0, v_{3s}) = s+1$, where $2 \leq s \leq m$.

From the above discussion, there exists j, where $6 \leq j \leq n/2$, such that $d(u_0, v_j) = \ell$ for any $3 \leq \ell < D$. Define the following sets of vertices:

$$
\begin{aligned}
V_1 &= \{u_i :\ 1 \leq i \leq j-1\} \cup \{v_i :\ 1 \leq i \leq j-1\},\\
V_2 &= \{u_i :\ j+1 \leq i \leq n-1\} \cup \{v_i :\ j+1 \leq i \leq n-1\},\\
W^1_{u_0v_j} &= W_{u_0v_j} \cup (V_1 \cup \{u_0, v_0, u_j, v_j\}),\\
W^1_{v_ju_0} &= W_{v_ju_0} \cup (V_1 \cup \{u_0, v_0, u_j, v_j\}),\\
W^2_{u_0v_j} &= W_{u_0v_j} \cup (V_2 \cup \{u_0, v_0, u_j, v_j\}),\\
W^2_{v_ju_0} &= W_{v_ju_0} \cup (V_2 \cup \{u_0, v_0, u_j, v_j\}).
\end{aligned}
$$

Because $6 \le j \le n/2$, we have $|W^2_{u_0v_j}| = |W^1_{u_0v_{n-j}}|$ and $|W^2_{v_ju_0}| = |W^1_{v_{n-j}u_0}|$. So

$$|W_{u_0v_j}| = |W^1_{u_0v_j}| + |W^2_{u_0v_j}| - 2 = |W^1_{u_0v_j}| + |W^1_{u_0v_{n-j}}| - 2 \quad \text{and}$$
$$|W_{v_ju_0}| = |W^1_{v_ju_0}| + |W^2_{v_ju_0}| - 2 = |W^1_{v_ju_0}| + |W^1_{v_{n-j}u_0}| - 2.$$

In the following we will compute $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$ where $6 \le j \le n-6$. The computation is divided into six cases, and for transparency and non-replication purposes, present only the first case in detail. Details for the other five cases are given in [19].

The computation of $|W^1_{u_0v_{3s}}|$ and $|W^1_{v_{3s}u_0}|$, where s is odd and $s \ge 5$, is as follows.

- If $0 \le t < s$, then $d(u_0, v_{3t}) = 1 + t$ and $d(v_{3s}, v_{3t}) = s - t$.
  If $0 \le t < \frac{s-1}{2}$, then $d(u_0, v_{3t}) < d(v_{3s}, v_{3t})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{3t}) > d(v_{3s}, v_{3t})$.
- If $0 \le t < s$, then $d(u_0, v_{3t+1}) = 2 + t$ and $d(v_{3s}, v_{3t+1}) = s - t + 3$.
  If $0 \le t < \frac{s+1}{2}$, then $d(u_0, v_{3t+1}) < d(v_{3s}, v_{3t+1})$.
  If $\frac{s+1}{2} < t < s$, then $d(u_0, v_{3t+1}) > d(v_{3s}, v_{3t+1})$.
- If $0 \le t < s$, then $d(u_0, v_{3t+2}) = 3 + t$ and $d(v_{3s}, v_{3t+2}) = s - t + 2$.
  If $0 \le t < \frac{s-1}{2}$, then $d(u_0, v_{3t+2}) < d(v_{3s}, v_{3t+2})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{3t+2}) > d(v_{3s}, v_{3t+2})$.
- If $1 \le t \le s$, then $d(u_0, u_{3t}) = 2 + t$ and $d(v_{3s}, u_{3t}) = s - t + 1$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{3t}) < d(v_{3s}, u_{3t})$.
  If $\frac{s-1}{2} < t \le s$, then $d(u_0, u_{3t}) > d(v_{3s}, u_{3t})$.
- If $1 \le t < s$, then $d(u_0, u_{3t+1}) = 3 + t$ and $d(v_{3s}, u_{3t+1}) = s - t + 2$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{3t+1}) < d(v_{3s}, u_{3t+1})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, u_{3t+1}) > d(v_{3s}, u_{3t+1})$.
- If $1 \le t < s$, then $d(u_0, u_{3t+2}) = 4 + t$ and $d(v_{3s}, u_{3t+2}) = s - t + 1$.
  If $1 \le t < \frac{s-3}{2}$, then $d(u_0, u_{3t+2}) < d(v_{3s}, u_{3t+2})$.
  If $\frac{s-3}{2} < t < s$, then $d(u_0, u_{3t+2}) > d(v_{3s}, u_{3t+2})$.
- $d(u_0, u_1) = 1$, $d(v_{3s}, u_1) = s + 2$, $d(u_0, u_2) = 2$, and $d(v_{3s}, u_2) = s + 1$.

Note that $u_0 \in W^1_{u_0v_{3s}}$ and $v_{3s} \in W^1_{v_{3s}u_0}$. Combined with the above discussion we obtain $|W^1_{u_0v_{3s}}| = 3s - 3$ and $|W^1_{v_{3s}u_0}| = 3s - 1$.

The conclusions in the remaining cases are as follows:

- If $s \ge 4$ and s is even, then $|W^1_{u_0v_{3s}}| = 3s$ and $|W^1_{v_{3s}u_0}| = 3s + 2$.

- If $s \geq 3$ and s is odd, then $|W^1_{u_0v_{3s+1}}| = 3s+1$ and $|W^1_{v_{3s+1}u_0}| = 3s+3$.
- If $s \geq 4$ and s is even, then $|W^1_{u_0v_{3s+1}}| = 3s-2$ and $|W^1_{v_{3s+1}u_0}| = 3s$.
- If $s \geq 5$ and s is odd, then $|W^1_{u_0v_{3s+2}}| = 3s-1$ and $|W^1_{v_{3s+2}u_0}| = 3s+1$.
- If $s \geq 4$ and s is even, then $|W^1_{u_0v_{3s+2}}| = 3s+2$ and $|W^1_{v_{3s+2}u_0}| = 3s+4$.
- $|W^1_{u_0v_6}| = 7$ and $|W^1_{v_6u_0}| = 7$.
- $|W^1_{u_0v_7}| = 6$ and $|W^1_{v_7u_0}| = 6$.
- $|W^1_{u_0v_8}| = 9$ and $|W^1_{v_8u_0}| = 9$.
- $|W^1_{u_0v_9}| = 7$ and $|W^1_{v_9u_0}| = 8$.
- $|W^1_{u_0v_{11}}| = 9$ and $|W^1_{v_{11}u_0}| = 10$.

When $n \geq 17$, from the above computation of $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$ where $6 \leq j \leq n-6$, for any $3 \leq \ell < D$, we know that there exists j where $d(u_0, v_j) = \ell$ and $6 \leq j \leq n/2$ such that $|W_{u_0v_j}| < |W_{v_ju_0}|$. □

3 Sketch proof of Theorem 4

As mentioned in the introduction, it suffices to prove that for any $n > 24$, the generalized Petersen graph $GP(n,4)$ is not ℓ-distance-balanced for any $1 \leq \ell < \mathrm{diam}(GP(n,4))$. We split the argument into the cases $\ell = 1$, $\ell = 2$, and $3 \leq \ell < \mathrm{diam}(GP(n,4))$ to be respectively covered by Propositions 8, 9, and 10.

Proposition 8. *For any $n > 24$, the generalized Petersen graph $GP(n,4)$ is not 1-distance-balanced.*

Proof. Since $d_{GP(n,4)}(u_0, v_0) = 1$, it suffices to prove that $|W_{u_0v_0}| < |W_{v_0u_0}|$. We divide the discussion into eight cases based on $n \bmod 8$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in [19].

Let $n = 8m$, where $m \geq 4$. By symmetry, it suffices to consider vertices u_i and v_i where $1 \leq i \leq \frac{n}{2}$. Then the following holds.

- If $1 \leq t \leq m$, then $d(u_0, v_{4t}) = 1 + t$ and $d(v_0, v_{4t}) = t$.
- If $0 \leq t < m$, then $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_0, v_{4t+1}) = 3 + t$.

- If $0 \le t < m$, then $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_0, v_{4t+2}) = 4 + t$.
- If $0 \le t < m$, then $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_0, v_{4t+3}) = 4 + t$.
- If $1 \le t \le m$, then $d(u_0, u_{4t}) = 2 + t$ and $d(v_0, u_{4t}) = 1 + t$.
- If $1 \le t < m$, then $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_0, u_{4t+1}) = 2 + t$.
- If $1 \le t < m$, then $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_0, u_{4t+2}) = 3 + t$.
- If $1 \le t < m$, then $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_0, u_{4t+3}) = 3 + t$.
- $d(u_0, u_1) = 1$, $d(v_0, u_1) = 2$, $d(u_0, u_2) = 2$, $d(v_0, u_2) = 3$, $d(u_0, u_3) = 3$, and $d(v_0, u_3) = 3$.

Note that $u_0 \in W_{u_0v_0}$ and $v_0 \in W_{v_0u_0}$. Combined with the above discussion we arrive at $|W_{u_0v_0}| = 2(3m+2)+1 = 6m+5$ and $|W_{v_0u_0}| = 2(5m-5)+3 = 10m-7$. Because $m \ge 4$, we can conclude that $|W_{u_0v_0}| < |W_{v_0u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 8m + 1$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 3$ and $|W_{v_0u_0}| = 10m - 5$.
- If $n = 8m + 2$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 4$ and $|W_{v_0u_0}| = 10m - 2$.
- If $n = 8m + 3$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 5$ and $|W_{v_0u_0}| = 10m - 1$.
- If $n = 8m + 4$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 8$ and $|W_{v_0u_0}| = 10m - 2$.
- If $n = 8m + 5$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 7$ and $|W_{v_0u_0}| = 10m + 1$.
- If $n = 8m + 6$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 6$ and $|W_{v_0u_0}| = 10m + 2$.
- If $n = 8m + 7$, where $m \ge 3$, then $|W_{u_0v_0}| = 6m + 7$ and $|W_{v_0u_0}| = 10m + 3$.

In each case we have $|W_{u_0v_0}| < |W_{v_0u_0}|$. □

Proposition 9. *For any $n > 24$, the generalized Petersen graph $GP(n, 4)$ is not 2-distance-balanced.*

Proof. Since $d_{GP(n,4)}(u_0, v_{-4}) = 2$, it suffices to prove that $|W_{u_0v_{-4}}| < |W_{v_{-4}u_0}|$. We divide the discussion into the eight cases based on $n \bmod 8$, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in [19].

Firstly we consider vertices $v_{-1}, v_{-2}, v_{-3}, u_{-1}, u_{-2}, u_{-3}$:

- $d(u_0, v_{-1}) = 2$ and $d(v_{-4}, v_{-1}) = 4$,
- $d(u_0, v_{-2}) = 3$ and $d(v_{-4}, v_{-2}) = 4$,
- $d(u_0, v_{-3}) = d(v_{-4}, v_{-3}) = 3$,
- $d(u_0, u_{-1}) = 1$ and $d(v_{-4}, u_{-1}) = 3$,
- $d(u_0, u_{-2}) = 2$ and $d(v_{-4}, u_{-2}) = 3$,
- $d(u_0, u_{-3}) = 3$ and $d(v_{-4}, u_{-3}) = 2$.

Next we consider vertices v_i, $0 \leq i < n-4$, and u_j, $1 \leq j \leq n-4$. Let $n = 8m$, $m \geq 4$. Note that $n - 4 = 8m - 4 = 4(2m-1)$.

- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t}) = d(v_{8m-4}, v_{4t}) = 1 + t$.
  If $m \leq t < 2m-1$, then $d(v_{8m-4}, v_{4t}) = 2m-t-1$ and $d(u_0, v_{4t}) > d(v_{8m-4}, v_{4t})$.
- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t+1}) = 2 + t$ and $d(u_0, v_{4t+1}) < d(v_{8m-4}, v_{4t+1})$.
  If $m \leq t < 2m-1$, then $d(u_0, v_{4t+1}) = d(v_{8m-4}, v_{4t+1}) = 2m - t + 2$.
- If $0 \leq t \leq m-1$, then $d(u_0, v_{4t+2}) = 3 + t$ and $d(u_0, v_{4t+2}) < d(v_{8m-4}, v_{4t+2})$.
  If $m \leq t < 2m-1$, then $d(u_0, v_{4t+2}) = d(v_{8m-4}, v_{4t+2}) = 2m - t + 2$.
- If $0 \leq t \leq m-2$, then $d(u_0, v_{4t+3}) = 3 + t$ and $d(u_0, v_{4t+3}) < d(v_{8m-4}, v_{4t+3})$.
  If $m-1 \leq t < 2m-1$, then $d(u_0, v_{4t+3}) = d(v_{8m-4}, v_{4t+3}) = 2m - t + 1$.
- If $1 \leq t \leq m-1$, then $d(u_0, u_{4t}) = d(v_{8m-4}, u_{4t}) = 2 + t$.
  If $m \leq t \leq 2m-1$, then $d(v_{8m-4}, u_{4t}) = 2m - t$ and $d(u_0, u_{4t}) > d(v_{8m-4}, u_{4t})$.
- If $1 \leq t \leq m-1$, then $d(u_0, u_{4t+1}) = d(v_{8m-4}, u_{4t+1}) = 3 + t$.
  If $m \leq t < 2m-1$, then $d(v_{8m-4}, u_{4t+1}) = 2m - t + 1$ and $d(u_0, u_{4t+1}) > d(v_{8m-4}, u_{4t+1})$.
- If $1 \leq t \leq m-2$, then $d(u_0, u_{4t+2}) = d(v_{8m-4}, u_{4t+2}) = 4 + t$.
  If $m-1 \leq t < 2m-1$, then $d(v_{8m-4}, u_{4t+2}) = 2m - t + 1$ and $d(u_0, u_{4t+2}) > d(v_{8m-4}, u_{4t+2})$.
- If $1 \leq t \leq m-2$, then $d(u_0, u_{4t+3}) = d(v_{8m-4}, u_{4t+3}) = 4 + t$.
  If $m-1 \leq t < 2m-1$, then $d(v_{8m-4}, u_{4t+3}) = 2m - t$ and $d(u_0, u_{4t+3}) > d(v_{8m-4}, u_{4t+3})$.
- $d(u_0, u_1) = 1$, $d(v_{8m-4}, u_1) = 2m + 1$, $d(u_0, u_2) = 2$, $d(v_{8m-4}, u_2) = 2m + 1$, $d(u_0, u_3) = 3$, and $d(v_{8m-4}, u_3) = 2m$.

Note that $u_0 \in W_{u_0 v_{8m-4}}$ and $v_{8m-4} \in W_{v_{8m-4} u_0}$. Combined with the above discussion we arrive at $|W_{u_0 v_{8m-4}}| = 3m + 7$ and $|W_{v_{8m-4} u_0}| = 5m$. Because $m \geq 4$ we may conclude that $|W_{u_0 v_{8m-4}}| < |W_{v_{8m-4} u_0}|$.

The conclusions in the remaining cases are as follows:

- If $n = 8m + 1$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 7$ and $|W_{v_{-4} u_0}| = 5m + 3$.
- If $n = 8m + 2$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 6$ and $|W_{v_{-4} u_0}| = 5m + 2$.
- If $n = 8m + 3$ where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 7$ and $|W_{v_{-4} u_0}| = 5m + 3$.
- If $n = 8m + 4$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 9$ and $|W_{v_{-4} u_0}| = 5m + 4$.
- If $n = 8m + 5$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 7$ and $|W_{v_{-4} u_0}| = 5m + 3$.
- If $n = 8m + 6$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 8$ and $|W_{v_{-4} u_0}| = 5m + 6$.
- If $n = 8m + 7$, where $m \geq 3$, then $|W_{u_0 v_{-4}}| = 3m + 8$ and $|W_{v_{-4} u_0}| = 5m + 6$.

In each case we have $|W_{u_0 v_{-4}}| < |W_{v_{-4} u_0}|$ as required. □

Proposition 10. *For any $n > 24$, the generalized Petersen graph $GP(n, 4)$ is not ℓ-distance-balanced for any $3 \leq \ell < \operatorname{diam}(GP(n, 4))$.*

Proof. For a given fixed n, we set $D = \operatorname{diam}(GP(n, 4))$.

For any $3 \leq \ell < D$, we first show that there exists v_j such that $d(u_0, v_j) = \ell$ where $8 \leq j \leq n/2$. From [18] we recall that there exists j^* such that $d(u_0, u_{j^*}) = D$.

If $n = 8m$, where $m \geq 4$, or $n = 8m + 1$, where $m \geq 3$, then from [18] we know that $j^* = 4(m - 1) + 2$ and $D = d(u_0, u_{j^*}) = m + 3$. Note that $d(u_0, v_{4s+2}) = s + 3$, where $2 \leq s \leq m - 1$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

If $n = 8m + 2$, where $m \geq 3$, or $n = 8m + 3$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 1$ and $D = d(u_0, u_{j^*}) = m + 3$. Note that $d(u_0, v_{4s+1}) = s + 2$, where $3 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

If $n = 8m + 4$, where $m \geq 3$, or $n = 8m + 5$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 2$ and $D = d(u_0, u_{j^*}) = m + 4$. Note that $d(u_0, v_{4s+2}) = s + 3$, where $2 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

If $n = 8m + 6$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 3$ and $D = d(u_0, u_{j^*}) = m + 4$. Note that $d(u_0, v_{4s+3}) = s + 3$, where $2 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

If $n = 8m + 7$, where $m \geq 3$, then from [18] we know that $j^* = 4m + 2$ and $D = d(u_0, u_{j^*}) = m + 4$. Note that $d(u_0, v_{4s+2}) = s + 3$, where $2 \leq s \leq m$, and $d(u_0, v_{4s}) = s + 1$, where $2 \leq s \leq m$.

By the above discussion, there exists j, where $8 \le j \le n/2$, such that $d(u_0, v_j) = \ell$ for any $3 \le \ell < D$. Define the following sets of vertices:

$$\begin{aligned}
V_1 &= \{u_i : \ 1 \le i \le j-1\} \cup \{v_i : \ 1 \le i \le j-1\}, \\
V_2 &= \{u_i : \ j+1 \le i \le n-1\} \cup \{v_i : \ j+1 \le i \le n-1\}, \\
W^1_{u_0v_j} &= W_{u_0v_j} \cup (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^1_{v_ju_0} &= W_{v_ju_0} \cup (V_1 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{u_0v_j} &= W_{u_0v_j} \cup (V_2 \cup \{u_0, v_0, u_j, v_j\}), \\
W^2_{v_ju_0} &= W_{v_ju_0} \cup (V_2 \cup \{u_0, v_0, u_j, v_j\}).
\end{aligned}$$

Because $8 \le j \le n/2$, we have $|W^2_{u_0v_j}| = |W^1_{u_0v_{n-j}}|$ and $|W^2_{v_ju_0}| = |W^1_{v_{n-j}u_0}|$. So

$$\begin{aligned}
|W_{u_0v_j}| &= |W^1_{u_0v_j}| + |W^2_{u_0v_j}| - 2 = |W^1_{u_0v_j}| + |W^1_{u_0v_{n-j}}| - 2 \quad \text{and} \\
|W_{v_ju_0}| &= |W^1_{v_ju_0}| + |W^2_{v_ju_0}| - 2 = |W^1_{v_ju_0}| + |W^1_{v_{n-j}u_0}| - 2.
\end{aligned}$$

In the following we will compute $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$ where $8 \le j \le n-8$. The computation is divided into eight cases, and for transparency and non-replication purposes, present only the first case in detail. Details for the other seven cases are given in [19].

The computation of $|W^1_{u_0v_{4s}}|$ and $|W^1_{v_{4s}u_0}|$, where $s \ge 5$ is odd is as follows.

- If $0 \le t < s$, then $d(u_0, v_{4t}) = 1 + t$ and $d(v_{4s}, v_{4t}) = s - t$.
  If $0 \le t < \frac{s-1}{2}$, then $d(u_0, v_{4t}) < d(v_{4s}, v_{4t})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{4t}) > d(v_{4s}, v_{4t})$.

- If $0 \le t < s$, then $d(u_0, v_{4t+1}) = 2 + t$ and $d(v_{4s}, v_{4t+1}) = s - t + 3$.
  If $0 \le t < \frac{s+1}{2}$, then $d(u_0, v_{4t+1}) < d(v_{4s}, v_{4t+1})$.
  If $\frac{s+1}{2} < t < s$, then $d(u_0, v_{4t+1}) > d(v_{4s}, v_{4t+1})$.

- If $0 \le t < s$, then $d(u_0, v_{4t+2}) = 3 + t$ and $d(v_{4s}, v_{4t+2}) = s - t + 3$.
  If $0 \le t \le \frac{s-1}{2}$, then $d(u_0, v_{4t+2}) < d(v_{4s}, v_{4t+2})$.
  If $\frac{s+1}{2} \le t < s$, then $d(u_0, v_{4t+2}) > d(v_{4s}, v_{4t+2})$.

- If $0 \le t < s$, then $d(u_0, v_{4t+3}) = 3 + t$ and $d(v_{4s}, v_{4t+3}) = s - t + 2$.
  If $0 \le t < \frac{s-1}{2}$, then $d(u_0, v_{4t+3}) < d(v_{4s}, v_{4t+3})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, v_{4t+3}) > d(v_{4s}, v_{4t+3})$.

- If $1 \le t \le s$, then $d(u_0, u_{4t}) = 2 + t$ and $d(v_{4s}, u_{4t}) = s - t + 1$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{4t}) < d(v_{4s}, u_{4t})$.
  If $\frac{s-1}{2} < t \le s$, then $d(u_0, u_{4t}) > d(v_{4s}, u_{4t})$.

- If $1 \le t < s$, then $d(u_0, u_{4t+1}) = 3 + t$ and $d(v_{4s}, u_{4t+1}) = s - t + 2$.
  If $1 \le t < \frac{s-1}{2}$, then $d(u_0, u_{4t+1}) < d(v_{4s}, u_{4t+1})$.
  If $\frac{s-1}{2} < t < s$, then $d(u_0, u_{4t+1}) > d(v_{4s}, u_{4t+1})$.

- If $1 \le t < s$, then $d(u_0, u_{4t+2}) = 4 + t$ and $d(v_{4s}, u_{4t+2}) = s - t + 2$.
  If $1 \le t \le \frac{s-3}{2}$, then $d(u_0, u_{4t+2}) < d(v_{4s}, u_{4t+2})$.
  If $\frac{s-1}{2} \le t < s$, then $d(u_0, u_{4t+2}) > d(v_{4s}, u_{4t+2})$.

- If $1 \le t < s$, then $d(u_0, u_{4t+3}) = 4 + t$ and $d(v_{4s}, u_{4t+3}) = s - t + 1$.
  If $1 \le t < \frac{s-3}{2}$, then $d(u_0, u_{4t+3}) < d(v_{4s}, u_{4t+3})$.
  If $\frac{s-3}{2} < t < s$, then $d(u_0, u_{4t+3}) > d(v_{4s}, u_{4t+3})$.

- $d(u_0, u_1) = 1$, $d(v_{4s}, u_1) = s+2$, $d(u_0, u_2) = 2$, $d(v_{4s}, u_2) = s+2$, $d(u_0, u_3) = 3$, and $d(v_{4s}, u_3) = s + 1$.

Note that $u_0 \in W^1_{u_0 v_{4s}}$ and $v_{4s} \in W^1_{v_{4s} u_0}$. Combined with the above discussion we arrive at $|W^1_{u_0 v_{4s}}| = 4s - 3$ and $|W^1_{v_{4s} u_0}| = 4s - 1$.

The conclusions in the remaining cases are as follows:

- If $s \ge 4$ and s is even, then $|W^1_{u_0 v_{4s}}| = 4s - 1$ and $|W^1_{v_{4s} u_0}| = 4s + 1$.
- If $s \ge 3$ and s is odd, then $|W^1_{u_0 v_{4s+1}}| = 4s + 1$ and $|W^1_{v_{4s+1} u_0}| = 4s + 3$.
- If $s \ge 4$ and s is even, then $|W^1_{u_0 v_{4s+1}}| = 4s - 3$ and $|W^1_{v_{4s+1} u_0}| = 4s - 1$.
- If $s \ge 5$ and s is odd, then $|W^1_{u_0 v_{4s+2}}| = 4s - 1$ and $|W^1_{v_{4s+2} u_0}| = 4s + 1$.
- If $s \ge 4$ and s is even, then $|W^1_{u_0 v_{4s+2}}| = 4s + 1$ and $|W^1_{v_{4s+2} u_0}| = 4s + 3$.
- If $s \ge 5$ and s is odd, then $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$.
- If $s \ge 4$ and s is even, then $|W^1_{u_0 v_{4s+3}}| = 4s + 1$ and $|W^1_{v_{4s+3} u_0}| = 4s + 3$.
- $|W^1_{u_0 v_8}| = 8$ and $|W^1_{v_8 u_0}| = 8$.
- $|W^1_{u_0 v_{10}}| = 11$ and $|W^1_{v_{10} u_0}| = 10$.
- $|W^1_{u_0 v_{11}}| = 10$ and $|W^1_{v_{11} u_0}| = 10$.
- $|W^1_{u_0 v_{12}}| = 10$ and $|W^1_{v_{12} u_0}| = 11$.
- $|W^1_{u_0 v_{14}}| = 12$ and $|W^1_{v_{14} u_0}| = 13$.
- $|W^1_{u_0 v_{15}}| = 14$ and $|W^1_{v_{15} u_0}| = 15$.

When $n \geq 26$, from the above computation of $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$, where $8 \leq j \leq n-8$, for any $3 \leq \ell < D$ we know that there exists j where $d(u_0, v_j) = \ell$ and $8 \leq j \leq n/2$ such that $|W_{u_0v_j}| < |W_{v_ju_0}|$. When $n = 25$, we have $d(u_0, v_8) = 3$, $d(u_0, v_{12}) = 4$, $d(u_0, v_{11}) = 5$, and $D(GP(25, 4)) = 6$. From the above computation of $|W^1_{u_0v_j}|$ and $|W^1_{v_ju_0}|$, we know that $|W_{u_0v_j}| < |W_{v_ju_0}|$ for any $j \in \{8, 11, 12\}$. □

4 Concluding remarks

In this paper, we prove that $GP(n, 3)$ is not ℓ-distance-balanced for $n > 16$ and $1 \leq \ell < \mathrm{diam}(GP(n, 3))$. We also prove that $GP(n, 4)$ is not ℓ-distance-balanced for $n > 24$ and $1 \leq \ell < \mathrm{diam}(GP(n, 4))$. Earlier it was proved in [21] that $GP(n, 2)$ is not ℓ-distance-balanced for $n > 11$ and $1 \leq \ell < \mathrm{diam}(GP(n, 2))$. As already mentioned, to investigate Conjecture 2 for $k \geq 5$, most likely a new approach is needed.

Having in mind Theorem 1, Conjecture 2, and the two main results of this paper, we propose the following problem.

Problem 11. *(1) Let $k \geq 5$ be odd and let $\frac{k(k+1)}{2} \leq n \leq (k+1)^2$. Determine whether $GP(n, k)$ is ℓ-distance-balanced for $1 \leq \ell < \mathrm{diam}(GP(n, k))$.*

(2) Let $k \geq 6$ be even and let $\frac{k^2}{2} \leq n \leq k(k+2)$. Determine whether $GP(n, k)$ is ℓ-distance-balanced for $1 \leq \ell < \mathrm{diam}(GP(n, k))$.

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Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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